Special Presymplectic Manifolds, Lagrangian submanifolds and the Lagrangian - Hamiltonian systems on Jet Bundles*

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Abstract

Special presymplectic manifolds are used to give a geometrical description of Lagrangian and Hamiltonian systems on Jet bundles in terms of Lagrangian submanifolds.

1 Introduction

The notion of special symplectic manifold was introduced by Tulczyjew around 1976 (see [18,19,14]). Special symplectic manifold theory gives a global description of Lagrangian and Hamiltonian formulations of Classical Mechanics. Indeed, a Hamiltonian (resp. Lagrange) function generates a Lagrangian submanifold of some symplectic tangent bundle and the local equations defining this submanifold are the Hamilton (resp., Euler-Lagrange) equations.

In a previous work [1,2] a geometrical picture of Lagrangian submanifolds in higher-order mechanical systems was presented. In this paper we shall continue the description of the above geometric formulation for first and higher-order Lagrangians in many independent variables. A higher-order Lagrangian L in many independent variables is defined on the tangent bundle of n^k-velocities, T^k_nQ of a given manifold Q, $1 \leq k < \infty$. If L depends explicitly on the independent variables $x \in R^n$, then L is defined on $R^n \times T^k_nQ$ (Lagrangians of such types are considered in physical field theories, for

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instance). We note that the process is different from that adopted for higher-order mechanics (i. e., $n = 1$), since we have no symplectic structures to use. Thus, we shall use presymplectic structures and introduce the notion of special presymplectic manifolds.

The paper is structured as follows. In section 2 we give some basic definitions and notations necessary to make the text comprehensible. In section 3 we recall some results for the ordinary situation. In section 4 we recall the notion of special presymplectic manifold introduced by Gotay and Nester [10]. In section 5 we examine the situation corresponding to first-order Lagrangians in many independent variables. As the process may be applied for Hamiltonians too we study the relation between both theories in section 6. Section 7 deals with higher-order cases and we describe the main differences with the first-order situation. We finish the work with the study of Lagrangians depending explicitly on the independent variables.

2 Preliminaries

All manifolds and mappings are supposed to be of class C^∞. In general the summation convention on repeated index is adopted.

Let M be a manifold of dimension m and R^n the Euclidean space with coordinates $(x) = (x_1, ..., x_n)$. Then $T_n^k M$ is the **tangent bundle of** n^k**- velocities** of M, i. e. , the manifold of all k- jets of mappings from R^n to M at the origin $0 \in R^n$. The manifold $T_n^k M$ is locally characterized as follows : if $(q) = (q^1, ..., q^m)$ are local coordinates for M then the coordinates $(q_\alpha) = (q_\alpha^i)$ are defined as follows, where $1 \leq i \leq m$ and $\alpha = (\alpha_1, ..., \alpha_n)$ is the multi-index of non-negative integers such that $\alpha_1 + ... + \alpha_n \leq k$,

$$q_\alpha(j_0^k \sigma) = \frac{1}{(\alpha_1)!...(\alpha_n)!} \frac{\partial^{\alpha_1+...+\alpha_n}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}} (q \circ \sigma)_{/x=0} \ ,$$

where $j_0^k \sigma$ is the k- jet at the origin $0 \in R^n$ of the map $\sigma : R^n \longrightarrow M$.

The **evolution tangent bundle of** n^k**- velocities** (for simplicity, the **evolution space**) is $R^n \times T_n^k M = J^k(R^n, M)$, the jet manifold of all mappings from R^n to M. Thus the local coordinates for $R^n \times T_n^k M$ are (x, q_α).

Now, let $T_n^* M$ be the manifold of 1- jets of all mappings from M to R^n with target at the origin $0 \in R^n$. Then we have $R^n \times T_n^* M = J^1(M, R^n)$.

In what follows we only consider the cases $k = 1$ or 2, since the general situation $2 < k < \infty$ only gives more complicated and tedious computations without introducing any new result. As we may writte $\alpha = (a_1) + (a_2) = (a_2) + (a_1), 1 \leq a_1, a_2 \leq n$, with $(a) = (0, ..., 0, 1, 0, ..., 0)$, where 1 is in the a- th position, the local coordinates for $T_n^1 M, T_n^* M, T_n^2 M, R^n \times T_n^1 M$ and $R^n \times T_n^2 M$ will be denoted by

$$(q, q_a), (q, p^a), (q, q_a, q_{ab}), (x, q, q_a), (x, q, q_a, q_{ab})$$

repectively, where $1 \leq a,b \leq n$ and $q_{ab} = q_{ba}$ (recall that we are omitting the index i).

Let $k = n = 1$. Then $T_1^1M = TM$ (resp. $T_1^*M = T^*M$) is the tangent (resp. cotangent) bundle of M. If $n = 1$, then $T_1^kM = T^kM$ is the tangent bundle of order k of M (see [3]). We denote by $\tau_M : TM \longrightarrow M$ (resp. $\bar{\tau}_M : T_n^1M \longrightarrow M$) and $\pi_M : T^*M \longrightarrow M$ (resp. $\bar{\pi}_M : T_n^*M \longrightarrow M$) for the bundle canonical projections. The Liouville canonical form θ_M on T^*M is the 1- form such that $d\theta_M = \omega_M$ is the canonical symplectic structure on T^*M, i. e. ,

$$< u, \theta_M(p) >=< T\pi_M(u), p >,$$

for any $u \in T_pT^*M, p \in T^*M$. Finally, we remark that a vector field on a manifold W with local coordinates (y) will be locally represented by $(y, \delta y)$ in place of $\delta y(\frac{\partial}{\partial y})$.

3 Lagrangian submanifolds and order-one mechanical systems

Let us recall the notion of special symplectic manifold (see [18,19,7]).

Definition 3.1 *A special symplectic manifold is a quintuple (X, M, π, λ, A) where $\pi : X \longrightarrow M$ is a fibre bundle, λ is a 1- form on X and $A : X \longrightarrow T^*M$ is a diffeomorphism such that $\pi = \pi_M \circ A$ and $\lambda = A^*\theta_M$.*

Now, let $F : M \longrightarrow R$ be a function. Then the set

$$N_F = \{z \in X / < u, \lambda >=< T\pi(u), dF >, \text{for any } u \in T_zX\} \tag{1}$$

is a Lagrangian submanifold of $(X, d\lambda)$, said to be **generated** by F. One has $N_F = (A^{-1} \circ dF)(M)$. The procedure is also valid when F is defined on a submanifold K of M, i. e. , the set

$$N_F = \{z \in X/\pi(z) \in K, \ < u, \lambda >=< T\pi(u), dF >,$$

$$\text{for any } u \in T_zX \text{ and } T\pi(u) \in T_zK\}, \tag{2}$$

is a Lagrangian submanifold of $(X, d\lambda)$.

Now, take $M = T^*Q$. Then the canonical symplectic form ω_Q on T^*Q induces a diffeomorphism $B_Q : TT^*Q \longrightarrow T^*T^*Q$ defined by

$$< u, B_Q(v) >= \omega_Q(v, u) ,$$

for any $u, v \in TT^*Q$. In local coordinates we have $B_Q(q, p, \dot{q}, \dot{p}) = (q, p, \dot{p}, -\dot{q})$. Since $\pi_{T^*Q} \circ B_Q = \tau_{T^*Q}$, if we define $\beta_Q = (B_Q)^*\theta_{T^*Q}$, then the quintuple

$$(TT^*Q, T^*Q, \tau_{T^*Q}, \beta_Q, B_Q)$$

is a special symplectic manifold. If $H : T^*Q \longrightarrow R$ is a Hamiltonian function on T^*Q, then the corresponding Hamiltonian vector field ξ_H is defined by $\xi_H = B_Q^{-1} \circ d(-H)$ or, equivalently, by $i_{\xi_H}\omega_Q = -dH$. In local coordinates we have

$$\xi_H(q,p) = (q, p, \frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q})$$

and the integral curves of ξ_H satisfy the Hamilton equations :

$$\dot{q} = \frac{\partial H}{\partial p}\ ,\ \dot{p} = -\frac{\partial H}{\partial q}$$

The Lagrangian submanifold of $(TT^*Q, d\beta_Q)$ generated by $-H$ is denoted by N_H and we have $N_H = \xi_H(T^*Q)$.

Next, take $M = TQ$. Then there exists a canonical diffeomorphism $A_Q : TT^*Q \longrightarrow T^*TQ$ such that $\pi_{TQ} \circ A_Q = T\pi_Q$; A_Q is locally given by $A_Q(q,p,\dot{q},\dot{p}) = (q,\dot{q},\dot{p},p)$. Thus, if we set $\alpha_Q = (A_Q)^*\theta_{TQ}$, we obtain a special symplectic manifold

$$(TT^*Q, TQ, T\pi_Q, \alpha_Q, A_Q)$$

One can check that the Lagrangian submanifold N_L of $(TT^*Q, d\alpha_Q)$ generated by the Lagrangian function $L : TQ \longrightarrow R$ is locally characterized by the Euler-Lagrange equations corresponding to L :

$$\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}}) = 0$$

Furthermore, we have $N_L = A_Q^{-1} \circ dL(TQ)$. If L is defined on some submanifold K of TQ then the Lagrangian submanifold N_L generated by L is locally characterized by the Euler-Lagrange equations with the constraints given by K.

In order to connect the Lagrangian submanifolds generated by $L : TQ \longrightarrow R$ and $H : T^*Q \longrightarrow R$ we proceed as follows. If $L : TQ \longrightarrow R$ is a Lagrangian function, we consider the Legendre transformation

$$Leg : TQ \longrightarrow T^*Q$$

locally defined by (see [7] for an intrinsical definition)

$$Leg(q, \dot{q}) = (q, p),$$

where $p = \partial L/\partial \dot{q}$. As we know, L is regular (resp. hyperregular) if and only if Leg is a local (resp. global) diffeomorphism. If L is hyperregular, then we define a Hamiltonian function H by $H \circ Leg = E_L$, where $E_L = p\dot{q} - L$. It is easy to see that $N_L = N_H$ (see [2] ; see also Section 6).

4 Special presymplectic manifolds

Definition 4.1 *A 2-form ω of constant rank $2r$ on a $(2r+s)$-dimensional manifold X is called a* **presymplectic** *form of rank $2r$ on X and the pair (X,ω) is said to be a* **presymplectic manifold.** *If $s=0$, then ω is a symplectic form and (X,ω) a symplectic manifold.*

Let Y be a submanifold of a presymplectic manifold (X,ω). The **symplectic complement** $TY^{\perp}$ of TY in TX is a vector bundle over Y, whose fiber at each $x \in Y$ is

$$(T_xY)^{\perp} = \{A \in T_xX / \omega(A,B) = 0, \text{for any} B \in T_xY\}$$

Then Y is said to be a **Lagrangian submanifold** of (X,ω) if $TY^{\perp} = TY$ (see [10,17]). Its dimension is $r+s$.

Next let us recall the notion of special presymplectic manifold [10].

Definition 4.2 *A* **special presymplectic manifold** *is a quintuple (X, M, π, λ, A) where $\pi : X \longrightarrow M$ be is a fibre bundle, λ is a 1- form on X and $A : X \longrightarrow T^*M$ is a submersion such that $\pi = \pi_M \circ A$ and $\lambda = A^*\theta_M$. It follows that $(X, d\lambda)$ is a presymplectic manifold with* rank $= 2(\dim M)$.

Indeed, if (x^i) are local coordinates for M, (x^i, u^r) are fibered coordinates for X and (x^i, p^i) are the induced coordinates on T^*M, then the Jacobian matrix for A is

$$B = \begin{pmatrix} I_m & \frac{\partial p_i}{\partial x^j} \\ 0 & \frac{\partial p_i}{\partial u^r} \end{pmatrix},$$

where $m = \dim M$. Since $A^*\lambda_M = \alpha$, the matrix associated to $d\alpha$ is

$$C = \begin{pmatrix} 0 & -\frac{\partial p_i}{\partial u^r} \\ \frac{\partial p_i}{\partial u^r} & 0 \end{pmatrix}.$$

Then rank $B = 2m$ if and only if rank $C = 2m$.

As $\lambda = A^*\theta_M$ implies that $A^*\omega_M = d\lambda$, then it is easy to see (like in the symplectic case) that if $F : M \longrightarrow R$ (resp. $F : K \subset M \longrightarrow R$) is a real function then the set N_F defined by (1) (resp. (2)) is a Lagrangian submanifold of the presymplectic manifold $(X, d\lambda)$. We say that N_F is **generated** by F.

Theorem 4.1 *Let Q be an m- dimensional manifold. Then there exists a canonical submersion $\bar{A}_Q : T^1_nT^*_nQ \longrightarrow T^*T^1_nQ$ such that the following diagram*

$$\begin{array}{ccc} T^1_nT^*_nQ & \xrightarrow{\bar{A}_Q} & T^*T^1_nQ \\ & {}_{T^1_n\pi_Q}\searrow \quad \swarrow_{\pi_{T^1_nQ}} & \\ & T^1_nQ & \end{array}$$

is commutative, i. e. ,

$$\pi_{T^1_nQ} \circ \bar{A}_Q = T^1_n \bar{\pi}_Q,$$

where $T^1_n\bar{\pi}_Q$ is the prolongation of $\bar{\pi}_Q : T^*_nQ \longrightarrow Q$ to $T^1_nT^*_nQ \longrightarrow T^1_nQ$.

Proof. First, let us recall that for any manifold M the bundle T^1_nM can be canonically identified as a vector bundle space over M with the Whitney sum $TM \oplus ... \oplus TM$ of TM with itself n times. In fact, let $v \in T^1_nM$. Then $v = j^1_0\sigma$ for some $\sigma : R^n \longrightarrow M$. Thus, if we define $\sigma_a : R \longrightarrow M$ by $\sigma_a(t) = \sigma(0, ..., 0, t, 0, ..., 0)$, where t is in the a-th position, $1 \leq a \leq n$, one obtains n tangent vectors $v_1, ..., v_n$ at the same point $\sigma(0) \in M$. Hence we writte $v = (v_1, ..., v_n)$. Locally, this identification is given by

$$(q, q_1, ..., q_n) = ((q, q_1), ..., (q, q_n)).$$

Also, we can identify the vector bundle T^*_nM with the Whitney sum $T^*M \oplus ... \oplus T^*M$ of T^*M with itself n times. If $\alpha = j^1_{(z,0)}f \in T^*_nM$, for some $f : M \longrightarrow R^n$ such that $f(z) = 0 \in R^n$, then we set $\alpha_a = j^1_{(z,0)}f^a$, where $f^a = pr_a \circ f : M \longrightarrow R$ and pr_a is given by $pr_a(x_1, ..., x_n) = x_a \in R$. Thus, one obtains n 1 -forms $\alpha_1, ..., \alpha_n$ at z, i. e. , $\alpha_a \in T^*_zM$, $1 \leq a \leq n$. In local coordinates the identification is given by

$$(q, p^1, ..., p^n) = ((q, p^1), ..., (q, p^n)).$$

Now, we can define $\bar{A}_Q$. Let $v \in T^1_n(T^*_nQ)$. Then $v \in (T^1_n)_\alpha(T^*_nQ)$ for some $\alpha \in T^*_nQ$, i. e. , $v = (v_1, ..., v_n)$, where $v_a \in T_\alpha(T^*_nQ), 1 \leq a \leq n$. As $\alpha \in T^*_nQ$, then $\alpha = (\alpha_1, ..., \alpha_n)$, with $\alpha_a \in T^*_qQ, q \in Q$. Because of the above identifications, if $v_a \in T_\alpha(T^*_nQ)$ then $v_a \in T_\alpha(T^*Q \oplus ... \oplus T^*Q) \cong T_{\alpha_1}(T^*Q) \oplus ... \oplus T_{\alpha_n}(T^*Q)$ and so $v_a = (v_{a1}, ..., v_{an})$, where $v_{ab} \in T_{\alpha_b}(T^*Q), 1 \leq b \leq n$, for each a. If $u \in T(T^1_nQ)$ then $u \in T_z(T^1_nQ) \cong T_z(TQ \oplus ... \oplus TQ) \cong T_{z_1}(TQ) \oplus ... \oplus T_{z_n}(TQ)$, where $z = (z_1, ..., z_n)$, $z_a \in T_qQ, q \in Q$. Thus, we may set $u = (u_1, ..., u_n)$ with $u_a \in T_{z_a}(TQ)$. Now, we define $\bar{A}_Q$ by

$$\langle u, \bar{A}_Q(v) \rangle = \sum_{a=1}^{n} \langle u_a, A_Q(v_{aa}) \rangle.$$

To end the proof, we shall express $\bar{A}_Q$ in local coordinates. We have the following bundle coordinates :

$$T^1_nQ : (q, q_1, ..., q_n)$$

$$T^*_nQ : (q, p^1, ..., p^n)$$

$$T^1_nT^*_nQ : (q, p^1, ..., p^n, q_{/1}, p^1_{/1}, ..., p^n_{/1}, ..., q_{/n}, p^1_{/n}, ..., p^n_{/n})$$

$$T^*T^1_nQ : (q, q_1, ..., q_n, \pi, \pi^1, ..., \pi^n)$$

So, if $v \in T^1_n T^*_n Q$, then we represent v by

$$v = (q, p^1, \dots, p^n, q_{/1}, p^1_{/1}, \dots, p^n_{/1}, \dots, q_{/n}, p^1_{/n}, \dots, p^n_{/n})$$

and for a, b fixed, we obtain

$$\begin{aligned} v_a &= (q, p^1, \dots, p^n, q_{/a}, p^1_{/a}, \dots, p^n_{/a}) \\ v_{ab} &= (q, p^b, q_{/a}, p^b_{/a}) \\ v_{aa} &= (q, p^a, q_{/a}, p^a_{/a}). \end{aligned}$$

On the other hand, if $u \in TT^1_n Q$, then we put $u = (q, q_1, \dots, q_n, r, s_1, \dots, s_n)$ and, for a fixed a, we have $u_a = (q, q_a, r, s_a)$.

Now, we set

$$\bar{A}_Q(v) = (q, q_1, \dots, q_n, \pi, \pi^1, \dots, \pi^n).$$

Since

$$\begin{aligned} \langle u_a, A_Q(v_{aa})\rangle &= \langle (q, q_a, r, s_a), A_Q(q, p^a, q_{/a}, p^a_{/a})\rangle \\ &= \langle (q, q_a, r, s_a), (q, q_{/a}, p^a_{/a}, p^a)\rangle \\ &= r p^a_{/a} + s_a p^a , \end{aligned}$$

we have

$$\langle u, \bar{A}_Q(v)\rangle = \sum_{a=1}^{n} (r p^a_{/a} + s_a p^a)$$

As $r, s_1, \dots, s_n$ are arbitrary we deduce

$$\pi = \sum_{a=1}^{n} p^a_{/a} \, , \; \pi^a = p^a \, , 1 \leq a \leq n \, .$$

Therefore

$$\begin{aligned} \bar{A}_Q(q, p^1, \dots, p^n, q_{/1}, p^1_{/1}, \dots, p^n_{/1}, \dots, q_{/n}, p^1_{/n}, \dots, p^n_{/n}) = \\ (q, q_{/1}, \dots, q_{/n}, \sum_{a=1}^{n} p^a_{/a}, p^1, \dots, p^n) \, . \end{aligned}$$

This shows that $\bar{A}_Q$ is a submersion and $\pi_{T^1_n Q} \circ \bar{A}_Q = T^1_n \bar{\pi}_Q$. □

Corollary 4.2 *The quintuple*

$$(T^1_n T^*_n Q, T^1_n Q, T^1_n \bar{\pi}_Q, \alpha_{T^1_n Q}, \bar{A}_Q),$$

where $\alpha_{T^1_n Q} = (\bar{A}_Q)^* \theta_{T^1_n Q}$ *is a special presymplectic manifold.*

5 Lagrangian submanifolds generated by a function $L : T^1_n Q \to R$

Let $L : T^1_n Q \to R$ be a Lagrangian function. Then L is a function depending on the variables $(q, q_a) = (q^i, q^i_1, ..., q^i_n)$, $1 \leq a \leq n$, $1 \leq i \leq m =$ dim Q , i. e. , L is an order-one Lagrangian in many independent variables. We set

$$N_L = \{z \in T^1_n T^*_n Q / < u, \alpha_{T^1_n Q} > = < TT^1_n \bar{\pi}_Q(u), dL >, \\ \text{for any } u \in T_z(T^1_n T^*_n Q)\}.$$

Then N_L is a Lagrangian submanifold of the presymplectic manifold

$$(T^1_n T^*_n Q, d\, \alpha_{T^1_n Q}).$$

The local equations defining N_L are obtained in the following manner. Let $u \in T_z(T^1_n T^*_n Q)$. Thus,

$$u = (q, p^1, ..., p^n, q_{/1}, p^1_{/1}, ..., p^n_{/1}, ..., q_{/n}, p^1_{/n}, ..., p^n_{/n}, \\ \delta q, \delta p^1, ..., \delta p^n, \delta q_{/1}, \delta p^1_{/1}, ..., \delta p^n_{/1}, ..., \delta q_{/n}, \delta p^1_{/n}, ..., \delta p^n_{/n})$$

Also,

$$\begin{aligned} \alpha_{T^1_n Q} &= (\bar{A}_Q)^* \theta_{T^1_n Q} = (\bar{A}_Q)^*(\pi dq + \pi^1 dq_1 + ... + \pi^n dq_n) \\ &= (\textstyle\sum_{a=1}^n p^a_{/a}) dq + p^1 dq_{/1} + ... + p^n dq_{/n} \\ TT^1_n \bar{\pi}_Q(u) &= (q, q_{/1}, ..., q_{/n}, \delta q, \delta q_{/1}, ..., \delta q_{/n}) \\ dL &= \tfrac{\partial L}{\partial q} dq + \tfrac{\partial L}{\partial q_1} dq_1 + ... + \tfrac{\partial L}{\partial q_n} dq_n \end{aligned}$$

Thus,

$$\begin{aligned} < u, \alpha_{T^1_n Q} > &= (\textstyle\sum_{a=1}^n p^a_{/a}) \delta q + p^1 \delta q_{/1} + ... + p^n \delta q_{/n} \\ < TT^1_n \bar{\pi}_Q(u), dL > &= \tfrac{\partial L}{\partial q} \delta q + \tfrac{\partial L}{\partial q_1} \delta q_{/1} + ... + \tfrac{\partial L}{\partial q_n} \delta q_{/n} \end{aligned}$$

from which we obtain

$$\frac{\partial L}{\partial q} = \sum_{a=1}^n p^a_{/a} \, , \, \frac{\partial L}{\partial q_1} = p^1 \, , ..., \frac{\partial L}{\partial q_n} = p^n \tag{3}$$

The Euler-Lagrange equations for a first-order Lagrangian $L : T^1_n Q \to R$ in many independent variables are :

$$\frac{\partial L}{\partial q} - \sum_{a=1}^{n} \frac{d}{dx_a}\left(\frac{\partial L}{\partial q_a}\right) = 0 \tag{4}$$

and the generalized momenta are defined by

$$p^a = \frac{\partial L}{\partial q_a} \tag{5}$$

Thus (3) is equivalent to (4) and (5).

An analogous result is obtained if the Lagrangian L is defined in a submanifold K of $T^1_n Q$. We have a Lagrangian submanifold N_L defined by

$$N_L = \{z \in T^1_n T^*_n Q / T^1_n \bar{\pi}_Q(z) \in K, \ < u, \alpha_{T^1_n Q} > = < TT^1_n \bar{\pi}_Q(u), dL >,$$
$$\text{for any } u \in T_z(T^1_n T^*_n Q) \text{ and } TT^1_n \bar{\pi}_Q(u) \in T_z K\},$$

which is locally characterized by the Euler-Lagrange equations with constraints.

EXAMPLE 1

Let us consider an example given by the Lagrangian density for a continous one-dimensional string

$$L = \frac{1}{2}\{\sigma(\frac{\partial y}{\partial t})^2 - \tau(\frac{\partial y}{\partial x})^2\} ,$$

where $y : R^2 \longrightarrow R$ is a real smooth function in the variables (t, x) and the coefficients σ and τ are constants. Then L is a real function defined on $T^1_2 R$ expressed in our notation by

$$L = \frac{1}{2}\{\sigma(q_1)^2 - \tau(q_2)^2\} ,$$

i. e. , $q = y$, $q_1 = y_t$, $q_2 = y_x$, where $y_t = \partial y/\partial t$, $y_x = \partial y/\partial x$. As N_L is characterized by the points of $T^1_2 T^*_2 R$ where

$$< u, \alpha_{T^1_2 R} > = < TT^1_2 \bar{\pi}_R(u), dL >$$

one obtains the following equalities :

$$p^1_{/1} + p^2_{/2} = 0, \ p^1 = \sigma q_1, \ p^2 = -\tau q_2 ,$$

which implies the well-known wave equation:

$$\sigma(\frac{\partial^2 y}{\partial t^2}) - \tau(\frac{\partial^2 y}{\partial x^2}) = 0$$

(see [13,12]).

6 Lagrangian submanifolds generated by a function $H : T^*_n Q \to R$

Let us now consider the jet bundle $T^*_n Q$. Then we define a canonical mapping

$$\bar{B}_Q : T^1_n(T^*_n Q) \longrightarrow T^*(T^*_n Q)$$

as follows : if $v = (v_1, \dots, v_n) \in (T^1_n)_\alpha(T^*_n Q)$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in T^*_n Q$ then $v_a = (v_{a1}, \dots, v_{an})$, $v_{ab} \in T_{\alpha_b}(T^*Q)$. Thus we have a canonical diffeomorphism $B_Q : T_{\alpha_b}(T^*Q) \longrightarrow T^*_{\alpha_b}(T^*Q)$ and we define

$$\langle X, \bar{B}_Q(v) \rangle = \sum_{a=1}^{n} \langle j^1_{0,\alpha_a}\gamma_a, B_Q(v_{aa}) \rangle ,$$

where $X = j^1_{0,\alpha}\gamma \in T_\alpha(T^*_n Q)$, $\gamma_a = \zeta_a \circ \gamma : R \longrightarrow T^*Q$ and $\zeta_a : T^*Q \oplus \dots \oplus T^*Q \longrightarrow T^*Q$ is the projection onto the a-th factor. Notice that $\zeta_a(\alpha) = \alpha_a$ and $B_Q(v_{aa}) \in T^*_{\alpha_a}(T^*Q)$.

If (q, p^a, Π, Π^a) are local coordinates for $T^*(T^*_n Q)$ then we obtain

$$\bar{B}_Q(q, p^1, \dots, p^n, q_{/1}, p^1_{/1}, \dots, p^n_{/1}, \dots, q_{/n}, p^1_{/n}, \dots, p^n_{/n}) = (q, p^1, \dots, p^n, \sum_{a=1}^{n} p^a_{/a}, -q_{/1}, \dots, -q_{/n})$$

Therefore, $\bar{B}_Q$ is a submersion such that the following diagram

$$\begin{array}{ccc} T^1_n(T^*_n Q) & \xrightarrow{\bar{B}_Q} & T^*(T^*_n Q) \\ & {\scriptstyle \bar{\tau}_{T^*_n Q}} \searrow \quad \swarrow {\scriptstyle \pi_{T^*_n Q}} & \\ & T^*_n Q & \end{array}$$

is commutative. Thus, the quintuple

$$(T^1_n(T^*_n Q), T^*_n Q, \bar{\tau}_{T^*_n Q}, \beta_{T^*_n Q}, \bar{B}_Q),$$

where $\beta_{T^*_n Q} = (\bar{B}_Q)^*\theta_{T^*_n Q}$, is a special presymplectic manifold.

Let $H : T^*_n Q \longrightarrow R$ be a Hamiltonian function. Then the Lagrangian submanifold N_H of $(T^1_n(T^*_n Q), \beta_{T^*_n Q})$ generated by $-H$ is locally defined by the following equations:

$$\frac{\partial q}{\partial x_a} = \frac{\partial H}{\partial p^a}, \quad \frac{\partial p^a}{\partial x_a} = -\frac{\partial H}{\partial q}$$

In order to establish a connection between the Lagrangian submanifolds generated by $L : T^1_n Q \to R$ and $H : T^*_n Q \longrightarrow R$ we proceed as follows. If $L : T^1_n Q \longrightarrow R$ is a Lagrangian function then we may consider the Legendre transformation induced by L

$$Leg : T^1_n Q \longrightarrow T^*_n Q$$

locally defined by (see [5] for an intrinsical definition)

$$Leg(q, q_a) = (q, \frac{\partial L}{\partial q_1}, \ldots, \frac{\partial L}{\partial q_n})$$

Like in the standard situation, we say that L is **regular** (resp. **hyperregular**) if Leg is a local (resp. global) diffeomorphism. Thus, if L is hyperregular, we define a Hamiltonian function H by $H \circ Leg = E_L$, where E_L is given by

$$E_L = \sum_{a=1}^{n} p^a q_a - L$$

(see [5]). We remark that if L is regular, then H is only locally defined. A simple computation shows that $N_L = N_H$.

Remark 1.- The reader can check that the Lagrangian density considered in Example 1 is hyperrregular.

7 The higher-order theory

Let $T^k_n Q$ be the tangent bundle of n^k -velocities of Q. As it is well-known there exists a canonical inclusion

$$j : T^k_n Q \longrightarrow T^1_n(T^{k-1}_n Q)$$

defined as follows. If $j^k_0\sigma \in T^k_n Q$, then $j(j^k_0\sigma) = j^1_0\tau$, where $\tau(x) = j^{k-1}_0\sigma_x$, $\sigma_x(y) = \sigma(x+y)$, $x, y \in R^n$.

From Theorem 1, there exists a canonical submersion

$$\bar{A}_{T^{k-1}_n Q} : T^1_n T^*_n(T^{k-1}_n Q) \longrightarrow T^* T^1_n(T^{k-1}_n Q)$$

Thus, we obtain a special presymplectic manifold

$$(T^1_n T^*_n(T^{k-1}_n Q), T^1_n(T^{k-1}_n Q), T^1_n \bar{\pi}_{T^{k-1}_n Q}, \alpha_{T^1_n(T^{k-1}_n Q)}, \bar{A}_{T^{k-1}_n Q})$$

Next, we consider the case $k = 2$. If $(q, q_a, q_{/b}, q_{a/b})$ are coordinates for $T^1_n(T^1_n Q)$ then the canonical inclusion

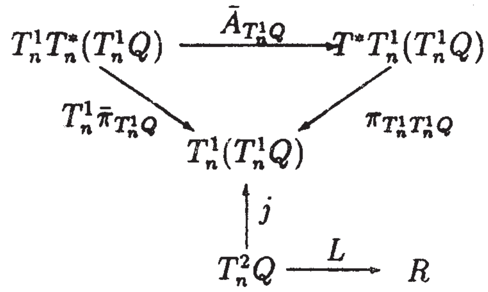
$$j : T^2_n Q \longrightarrow T^1_n(T^1_n Q)$$

is locally given by

$$j(q, q_a, q_{ab}) = (q, q_a, q_{/_b}, q_{a_{/_b}}),$$

where $q_{/_b} = q_b, q_{a_{/_b}} = q_{ab}$.

Now, let $L : T_n^2 Q \longrightarrow R$ be a Lagrangian of order 2. Then we have the following illustrative diagram :



Let N_L be the Lagrangian submanifold generated by L of the presymplectic manifold $(T_n^1 T_n^*(T_n^1 Q), d\alpha_{T_n^1 T_n^1 Q})$. We shall compute the local equations for N_L. We have the following bundle coordinates :

$$T_n^1 Q : (q, q_a)$$

$$T_n^*(T_n^1 Q) : (q, q_a, p^1, p_a^1, \ldots, p^n, p_a^n)$$

$$T_n^1 T_n^*(T_n^1 Q) : (q, q_a, p^1, p_a^1, \ldots, p^n, p_a^n, q_{/_b}, q_{a_{/_b}}, p^1_{/_b}, p^1_{a_{/_b}}, \ldots, p^n_{/_b}, p^n_{a_{/_b}})$$

$$T_n^1 T_n^1 Q : (q, q_a, q_{/_b}, q_{a_{/_b}})$$

$$T^* T_n^1(T_n^1 Q) : (q, q_a, q_{/_b}, q_{a_{/_b}}, \Pi, \Pi^a, \Pi^{0b}, \Pi^{ab})$$

The Lagrangian submanifold N_L is defined by

$$N_L = \{z \in T_n^1 T_n^*(T_n^1 Q) / T_n^1 \bar{\pi}_{T_n^1 Q}(z) \in j(T_n^2 Q),$$
$$\langle u, \alpha_{T_n^1 T_n^1 Q} \rangle = \langle T T_n^1 \bar{\pi}_{T_n^1 Q} u, dL \rangle, \text{for any } u \in T_z(T_n^1 T_n^*(T_n^1 Q))$$
$$\text{and } T_n^1 \bar{\pi}_{T_n^1 Q}(u) \in Tj(T T_n^2 Q)\}$$

Since

$$\bar{A}_{T_n^1 Q}(q, q_a, p^1, p_a^1, \ldots, p^n, p_a^n; q_{/_b}, q_{a_{/_b}}, p^1_{/_b}, p^1_{a_{/_b}}, \ldots, p^n_{/_b}, p^n_{a_{/_b}})$$
$$= (q, q_a, q_{/_a}, q_{b_{/_a}}, \Pi, \Pi^a, \Pi^{0b}, \Pi^{ab}),$$

where

$$q_{/a} = q_a, q_{b/a} = q_{ba}, \Pi = \sum_{a=1}^{n} p^a_{/a}, \Pi^a = \sum_{b=1}^{n} p^b_{a/b}, \Pi^{0b} = p^a, \Pi^{ab} = p^b_a$$

and

$$\theta_{T_n^*(T_n^1 Q)} = \Pi dq + \Pi^a dq_a + \Pi^{0b} dq_{/b} + \Pi^{ab} dq_{a/b} ,$$

then we deduce

$$\alpha_{T_n^1 T_n^1 Q} = (\sum_{a=1}^{n} p^a_{/a}) dq + (\sum_{b=1}^{n} p^b_{a/b} + \sum_{a=1}^{n} p^a) dq_{/a} + \sum_{a,b=1}^{n} p^a_b dq_{b/a}$$

Now suppose that

$$z = (q, q_a, p^1, p^1_a, ..., p^n, p^n_a; q_{/b}, q_{a/b}, p^1_{/b}, p^1_{a/b}, ..., p^n_{/b}, p^n_{a/b}) ,$$

$$u = (q, q_a, p^1, p^1_a, ..., p^n, p^n_a; q_{/b}, q_{a/b}, p^1_{/b}, p^1_{a/b}, ..., p^n_{/b}, p^n_{a/b};$$

$$\delta q, \delta q_a, \delta p^1, \delta p^1_a, ..., \delta p^n, p^n_a; \delta q_{/b}, \delta q_{a/b}, \delta p^1_{/b}, \delta p^1_{a/b}, ..., \delta p^n_{/b}, \delta p^n_{a/b}).$$

Since $T_n^1 \bar{\pi}_{T_n^1 Q}(z) \in j(T_n^2 Q)$, we deduce that $q_{/b} = q_b$, $q_{a/b} = q_{ab}$. **Furthermore,** $TT_n^1 \bar{\pi}_{T_n^1 Q}(u) \in Tj(TT_n^2 Q)$ implies that $\delta q_{/b} = \delta q_b$, $\delta q_{a/b} = \delta q_{ab}$.

Then

$$\langle u, \alpha_{T_n^1 T_n^1 Q} \rangle = (\sum_{a=1}^{n} p^a_{/a}) \delta q + (\sum_{b=1}^{n} p^b_{a/b} + \sum_{a=1}^{n} p^a) \delta q_{/a} + (\sum_{a,b=1}^{n} p^a_b) \delta q_{b/a}$$

On the other hand, we obtain

$$\langle TT_n^1 \bar{\pi}(u), dL \rangle = (\frac{\partial L}{\partial q}) \delta q + (\frac{\partial L}{\partial q_a}) \delta q_a + (\frac{\partial L}{\partial q_{ab}}) \delta q_{ab} .$$

Hence we have

$$\frac{\partial L}{\partial q} = \sum_{a=1}^{n} p^a_{/a} \ , \frac{\partial L}{\partial q_a} = \sum_{b=1}^{n} p^b_{a/b} + p^a \ , \frac{\partial L}{\partial q_{ab}} = p^a_b$$

The Euler-Lagrange equations for L are :

$$\frac{\partial L}{\partial q} - \sum_{a=1}^{n} \frac{d}{dx_a}(\frac{\partial L}{\partial q_a}) + \sum_{a,b=1}^{n} \frac{d^2}{dx_a dx_b}(\frac{\partial L}{\partial q_{ab}}) = 0$$

with

$$p_b^a = \frac{\partial L}{\partial q_{ab}}, p^a = \frac{\partial L}{\partial q_a} - \sum_{b=1}^{n} \frac{d}{dx_b}(\frac{\partial L}{\partial q_{ab}})$$

Therefore

$$\begin{aligned} \frac{\partial L}{\partial q} &= \sum_{a=1}^{n} \frac{d}{dx_a}(\frac{\partial L}{\partial q_a} - \frac{d}{dx_b}(\frac{\partial L}{\partial q_{ab}})) \\ &= \sum_{a=1}^{n} \frac{d}{dx_a} p^a = \sum_{a=1}^{n} p^a_{/a} \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \frac{\partial L}{\partial q_a} &= \sum_{b=1}^{n} \frac{d}{dx_b}(\frac{\partial L}{\partial q_{ab}} + p^a) \\ &= \sum_{b=1}^{n} \frac{d}{dx_b} p_b^a + p^a = \sum_{b=1}^{n} p^b_{a/_b} + p^a \end{aligned}$$

Consequently, the local equations defining N_L are precisely the Euler-Lagrange equations for L.

EXAMPLE 2

Let us consider the Lagrangian density for the free vibration of a beam

$$L = \frac{1}{2}\{\sigma(\frac{\partial y}{\partial t})^2 - \tau(\frac{\partial^2 y}{\partial^2 x})^2\},$$

where $y : R^2 \longrightarrow R$ is a real smooth function in the variables (t, x) and σ and τ are constants. Then $L : T_2^2 R \longrightarrow R$ is expressed in our notation by

$$L = \frac{1}{2}\{\sigma(q_1)^2 - \tau(q_{22})^2\},$$

i. e. ,

$$\begin{aligned} q = y, q_1 = y_t = \partial y/\partial t, q_2 = y_x = \partial y/\partial x, q_{11} = y_{tt} = \partial^2 y/\partial t^2, \\ q_{12} = q_{21} = y_{tx} = \partial^2 y/\partial t \partial x, q_{22} = y_{xx} = \partial^2 y/\partial x^2. \end{aligned}$$

One easily deduces that N_L is a Lagrangian submanifold of the presymplectic manifold $(T_2^1 T_2^* T_2^1 R, d\alpha_{T_2^1 T_2^1 R})$ locally characterized by

$$\begin{aligned} p^1_{/1} + p^2_{/2} = 0, p^1_{1/_1} + p^1_{1/_2} = \sigma q_1, p^1_{2/_1} + p^1_{2/_2} = 0, \\ p_1^1 = p_1^2 = p_2^1 = 0, p_2^2 = -\tau q_{22}, \end{aligned}$$

which implies the Euler-Lagrange equations :

$$\sigma\frac{\partial^2 L}{\partial t^2} - \tau\frac{\partial^4 L}{\partial x^4} = 0$$

(see [12]).

In higher order mechanics ($n = 1,\ k \geq 1$) [3] and in first order field theories ($n > 1,\ k = 1$) [14], one has both Lagrangian and Hamiltonian descriptions, which may be equivalent. In fact, if the Lagrangian is regular, the corresponding Lagrangian submanifolds can be identified (see [2], for the higher order mechanics case). However, in higher order field theories, that is for $n > 1,\ k > 2$, there is not an appropriate Hamiltonian description. This problem has been the goal of many researchers (see [3,4,4,6,8,9,11,15,16]).

8 The theory for order-one evolution spaces

Assume that $L : R^n \times T^1_n Q \longrightarrow R$ is an order-one Lagrangian depending explicitly on the independent variables $x \in R^n$. We consider the canonical inclusion

$$i : R^n \times T^1_n Q \longrightarrow T^1_n(R^n \times Q)$$

defined by

$$i(x, j^1_0\sigma) = j^1_0\bar{\sigma},\ \bar{\sigma}(x) = (x, \sigma(x)) .$$

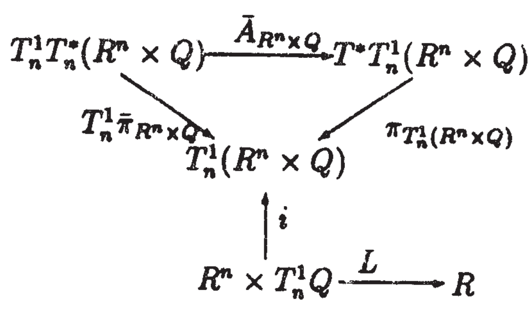
Hence i is locally given by

$$i(x_a, q, q_b) = (x_a, q, x_{ab}, q_b),$$

where $x_{ab} = \delta^b_a$. From Theorem 1, we have a special presymplectic manifold

$$(T^1_n T^*_n(R^n \times Q), T^1_n(R^n \times Q), T^1_n\bar{\pi}_{R^n\times Q}, \alpha_{T^1_n(R^n\times Q)}, \bar{A}_{R^n\times Q}) .$$

The following diagram illustrates the situation :



Next, we shall compute the local equations of the Lagrangian submanifold N_L of $(T^1_n T^*_n(R^n \times Q),\ d\alpha_{T^1_n(R^n\times Q)})$ generated by L. We have the following bundle coordinates :

$$T^1_n(R^n \times Q) : (x_a, q, x_{ab}, q_b)$$
$$T^*_n(R^n \times Q) : (x_a, q, (p^x)^b_a, p^b)$$
$$T^1_n T^*_n(R^n \times Q) : (x_a, q, (p^x)^b_a, p^b, x_{a_{/c}}, q_c, (p^x)^b_{a_{/c}}, p^b_{/c})$$
$$T^* T^1_n(R^n \times Q) : (x_a, q, x_{ab}, q_b, (\Pi^x)^a, \Pi, (\Pi^x)^{ab}, \Pi^b)$$

Remark 2.- The superscript x means that these p's are related to the independent variables when a Lagrangian L is considered - see below.

Now, a straightforward computation shows that the local equations for N_L are

$$\frac{\partial L}{\partial x_a} = \sum_{b=1}^{n} (p^x)^b_{a_{/b}} \ , \frac{\partial L}{\partial q} = \sum_{b=1}^{n} p^b_{/b} \ , \frac{\partial L}{\partial q_a} = p^a$$

Hence we have

$$\frac{\partial L}{\partial q} = \sum_{b=1}^{n} p^b_{/b} = \sum_{b=1}^{n} \frac{d}{dx_b}(\frac{\partial L}{\partial q_b})$$

which implies

$$\frac{\partial L}{\partial q} - \sum_{b=1}^{n} \frac{d}{dx_b}(\frac{\partial L}{\partial q_b}) = 0 \ ,$$

the Euler-Lagrange equations for L.

EXAMPLE 1 (continued)

The string's Lagrangian density seen in Example 1 gives an example for the present situation, if we consider the coefficients σ and τ as functions depending in the variable $x \in R$, i. e. , L is now a function defined on the submanifold $R \times T^1_2 R \cong R \times \{0\} \times T^1_2 R$ of the evolution space $R^2 \times T^1_2 R$.

Let $H : R^n \times T^*_n Q \longrightarrow R$ be a Hamiltonian function. We can consider the extended phase space $T^*_n(R^n \times Q)$ and define a canonical projection $\rho : T^*_n(R^n \times Q) \longrightarrow R^n \times T^*_n Q$ by

$$\rho(j^1_{(x,q),0} f) = (x, j^1_{q,0} \bar{f}) \ ,$$

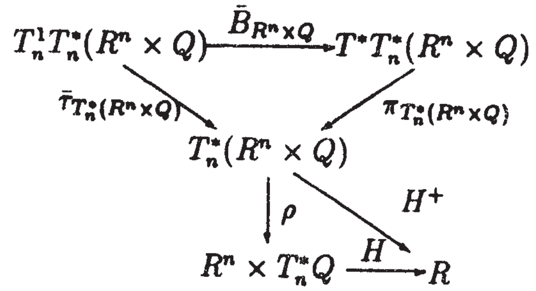
where $\bar{f}(q) = f(0, q)$, $q \in Q$. If $(x_a, q, (p^x)^b_a, p^b)$ are local coordinates for $T^*_n(R^n \times Q)$ then we have

$$\rho(x_a, q, (p^x)_a^b, p^b) = (x_a, q, p^b)$$

Now we define an extended Hamiltonian $H^+ : T_n^*(R^n \times Q) \longrightarrow R$ as follows :

$$H^+(x_a, q, (p^x)_a^b, p^b) = H(x_a, q, p^b) + \sum_{b=1}^{n} (p^x)_b^b$$

and we obtain the following diagram :



Then we have a Lagrangian submanifold N_{H^+} generated by $-H^+$ of the presymplectic manifold

$$(T_n^1 T_n^*(R^n \times Q), d\,\beta_{T_n^*(R^n \times Q)}) \, ,$$

where

$$\beta_{T_n^*(R^n \times Q)} = (\bar{B}_{R^n \times Q})^* \theta_{T_n^*(R^n \times Q)}$$

Furthermore, the Hamilton equations for H^+ are precisely the local equations defining N_{H^+} :

$$\frac{\partial q}{\partial x_a} = \frac{\partial H}{\partial p^a}, \; \frac{\partial p^a}{\partial x_a} = -\frac{\partial H}{\partial q}, \; \frac{\partial (p^x)_a^b}{\partial x_b} = -\frac{\partial H}{\partial x_a}$$

The Legendre transformation defined by L (see [5])

$$Leg : R^n \times T_n^1 Q \longrightarrow R^n \times T_n^* Q$$

is locally given by

$$Leg(x_a, q, q_a) = (x_a, q, \frac{\partial L}{\partial q_1}, \ldots, \frac{\partial L}{\partial q_n}) \, .$$

If we define $H : R^n \times T_n^* Q \longrightarrow R$ as in section 6 and H^+ as above, then a simple computation in local coordinates shows that $N_L = N_{H^+}$.

9 The theory for higher-order evolution spaces

Let $k = 2$ and$L : R^n \times T^2_n Q \longrightarrow R$ a Lagrangian function depending explicitly on the independent variables $x \in R^n$. Then we consider the canonical inclusion

$$R^n \times T^2_n Q \xrightarrow{\;j\;} R^n \times T^1_n(T^1_n Q) \xrightarrow{\;i\;} T^1_n(R^n \times T^1_n Q), \qquad \iota = i \circ j$$

locally given by

$$\iota(x_a, q, q_a, q_{ab}) = (x_a, q, q_a, x_{ab}, q_b, q_{ab}),$$

where $x_{ab} = \delta^b_a$. Then we obtain the following diagram :

$$\begin{array}{ccc}
T^1_n T^*_n(R^n \times T^1_n Q) & \xrightarrow{\bar{A}_{R^n \times T^1_n Q}} & T^* T^1_n(R^n \times T^1_n Q) \\
& {\scriptstyle T^1_n \bar{\pi}_{R^n \times T^1_n Q}} \searrow \quad \swarrow {\scriptstyle \pi_{T^1_n(R^n \times T^1_n Q)}} & \\
& T^1_n(R^n \times T^1_n Q) & \\
& \uparrow \iota & \\
& R^n \times T^2_n Q \xrightarrow{\;L\;} R &
\end{array}$$

Thus, we have a special presymplectic manifold

$$(T^1_n T^*_n(R^n \times T^1_n Q), T^1_n(R^n \times T^1_n Q), T^1_n \bar{\pi}_{R^n \times T^1_n Q}, \alpha_{T^1_n(R^n \times T^1_n Q)}, \bar{A}_{R^n \times T^1_n Q}) \; ,$$

where $\alpha_{T^1_n(R^n \times T^1_n Q)} = (\bar{A}_{R^n \times T^1_n Q})^* \theta_{T^1_n(R^n \times T^1_n Q)}$. Next, we shall compute the local equations of the Lagrangian submanifold N_L generated by L of the presymplectic manifold

$$(T^1_n T^*_n(R^n \times T^1_n Q), d\, \alpha_{T^1_n(R^n \times T^1_n Q)})$$

We have the following bundle coordinates :

$$T^1_n T^*_n(R^n \times T^1_n Q) : (x_a, q, q_a, (p^x)^b_a, p^b, p^b_a, x_{a/c}, q_c, q_{a/c}, (p^x)^b_{a/c}, p^b_{/c}, p^b_{a/c})$$
$$T^* T^1_n(R^n \times T^1_n Q) : (x_a, q, q_a, x_{a/b}, q_{/b}, q_{a/b}, (\Pi^x)^a, \Pi, \Pi^a, (\Pi^x)^{ab}, \Pi^b, \Pi^{ab})$$

Thus, we obtain the local equations for N_L :

$$\frac{\partial L}{\partial x_a} = \sum_{b=1}^{n}(p^x)^b_{a/b} \ , \frac{\partial L}{\partial q} = \sum_{b=1}^{n} p^b_{/b} \ , \frac{\partial L}{\partial q_a} = \sum_{b=1}^{n}(p)^b_{a/b} + p^a \quad \frac{\partial L}{\partial q_{ab}} = p^b_a$$

which implies the Euler-Lagrange equations for L :

$$\frac{\partial L}{\partial q} - \sum_{a=1}^{n} \frac{d}{dx_a}(\frac{\partial L}{\partial q_a}) + \sum_{a,b=1}^{n} \frac{d^2}{dx_a dx_b}(\frac{\partial L}{\partial q_{ab}}) = 0$$

EXAMPLE 2 (continued)

If we consider σ and τ as functions depending on the variable $x \in R$, then L will be a function defined on the submanifold $R \times T^2_2 R$ of the evolution space $R^2 \times T^2_2 R$.

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